Arthur Prior’s temporal logic and the origin of contemporary hybrid logic

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Plan of talk

I Ordinary tense logic  
(one language for talking about time)

II Translation to first-order logic  
(another language for talking about time)

III Arthur Prior’s philosophical problem

IV Prior’s solution: Hybrid tense logic

V Did Prior reach his philosophical goal?

VI The development of hybrid logic since Prior
Ordinary tense logic
(one language for talking about time)
A.J.P. Kenny summed up Prior’s life and work as follows.

Prior’s greatest scholarly achievement was undoubtedly the creation and development of tense-logic.

He had many different interests at different periods of his life, but from different angles he constantly returned to the same central and unchanging themes. Throughout his life, for instance, he worked away at the knot of problems surrounding determinism: first as a predestinarian theologian, then as a moral philosopher, finally as a metaphysician and logician.
Arthur Prior
The founding father of tense logic
and what is now called hybrid logic
Background: Two views on time (McTaggart 1908)

A-view (internal, dynamic)  B-view (external, static)
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A-view (internal, dynamic)  
Flow of time experienced by humans

B-view (external, static)  
Gods-eye-view of time
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<td>Formalization: First-order logic Variables ranging over times The earlier-later relation</td>
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Ordinary tense logic

Ordinary tense logic is obtained by extending propositional logic with the tense operators

\( F \) (symbolizing ”it will be the case that”)

and

\( P \) (symbolizing ”it was the case that”).
The language of tense logic, formally

The formal language of tense logic is generated by the grammar

$$\phi ::= p \mid \phi \land \phi \mid \neg \phi \mid F \phi \mid P \phi$$

where $p$ is a propositional symbol.

Note that $\land$ and $\neg$ are standard connectives of propositional logic.
The formal language of tense logic is generated by the grammar

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Note that $\land$ and $\neg$ are standard connectives of propositional logic.

**Note:**
Past, present and future in human languages are A-time
Example of A-time

Jais Nielsen
”Departure!”
1918
Formal (Kripke) semantics of tense logic

A \textit{frame} for tense logic is a pair \((T, <)\) where

\begin{itemize}
  \item \(T\) is a non-empty set; and
  \item \(<\) is a binary relation on \(T\).
\end{itemize}

The elements of \(T\) are called \textit{instants, moments}, or \textit{times} and the relation \(<\) is called the \textit{earlier-later relation}. 
A *frame* for tense logic is a pair \((T, <)\) where

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The elements of \(T\) are called *instants*, *moments*, or *times* and the relation \(<\) is called the *earlier-later relation*.

A *model* for tense logic is a triple \((T, <, V)\) where

- \((T, <)\) is a frame; and
- \(V\) is a function that to a moment and a propositional symbol assigns an element of \(\{0, 1\}\).

The function \(V\) is called a *valuation*. 
Given a model $\mathcal{M} = (T, <, V)$, the relation $\mathcal{M}, t \models \phi$ is defined by induction, where $t$ is an element of $T$ and $\phi$ is a formula.

- $\mathcal{M}, t \models p$ iff $V(t, p) = 1$
- $\mathcal{M}, t \models \phi \land \psi$ iff $\mathcal{M}, t \models \phi$ and $\mathcal{M}, t \models \psi$
- $\mathcal{M}, t \models \neg \phi$ iff not $\mathcal{M}, t \models \phi$
- $\mathcal{M}, t \models F\phi$ iff for some $u \in T$ such that $t < u$, it is the case that $\mathcal{M}, u \models \phi$
- $\mathcal{M}, t \models P\phi$ iff for some $u \in T$ such that $u < t$, it is the case that $\mathcal{M}, u \models \phi$

A formula $\phi$ is true at $t$ if $\mathcal{M}, t \models \phi$; otherwise it is false at $t$. 
Formal (Kripke) semantics of tense logic, continued

Given a model $\mathcal{M} = (T, <, V)$, the relation $\mathcal{M}, t \models \phi$ is defined by induction, where $t$ is an element of $T$ and $\phi$ is a formula.

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\mathcal{M}, t \models F \phi & \iff \text{for some } u \in T \text{ such that } t < u, \text{ it is the case that } \mathcal{M}, u \models \phi \\
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\end{align*}
$$

A formula $\phi$ is true at $t$ if $\mathcal{M}, t \models \phi$; otherwise it is false at $t$.

Note that the truth-value of a formula is relative to a moment, that is, a formula is evaluated "locally".
The operators $F$ and $P$ have duals $G$ and $H$ which are defined as respectively $\neg F \neg$ and $\neg P \neg$.

It follows that:

$$M, t \models G\phi \iff \text{for all } u \in T \text{ such that } t < u, \text{ it is the case that } M, u \models \phi$$

$$M, t \models H\phi \iff \text{for all } u \in T \text{ such that } u < t, \text{ it is the case that } M, u \models \phi$$

The operator $G$ symbolizes ”it is always going to be the case that” and $H$ symbolizes ”it has always been the case that”.
Part II

Translation to first-order logic
(another language for talking about time)
The first-order logic under consideration

Important observation:

A tense-logical model \((T, <, V)\) corresponds to a first-order model for the first-order language generated by the grammar

\[
\phi ::= p^*(a) \mid a < b \mid a = b \mid \phi \land \phi \mid \neg \phi \mid \forall a \phi
\]

where \(p\) is a propositional symbol of tense logic, and \(a\) and \(b\) are first-order variables.

Prior called it \emph{first-order earlier-later logic}. 
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Note:
History books and chronological tables involve B-time
Example of B-time

Chronological table from the historical journal Skalk
The *standard translation* to first-order logic

The translations $ST_a$ and $ST_b$ are defined by mutual induction.

\[
\begin{align*}
ST_a(p) &= p^*(a) \\
ST_a(\phi \land \psi) &= ST_a(\phi) \land ST_a(\psi) \\
ST_a(\neg \phi) &= \neg ST_a(\phi) \\
ST_a(F\phi) &= \exists b(a < b \land ST_b(\phi)) \\
ST_a(P\phi) &= \exists b(b < a \land ST_b(\phi))
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The definition of $ST_b$ is obtained by exchanging $a$ and $b$. 
The standard translation to first-order logic

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The definition of $ST_b$ is obtained by exchanging $a$ and $b$.

An assignment is a function from first-order variables to moments. Let a model $\mathcal{M} = (T, <, V)$ and an assignment $g$ be given.

**Theorem**

For any modal-logical formula $\phi$, it is the case that $\mathcal{M}, g(a) \models \phi$ if and only if $\mathcal{M}, g \models ST_a(\phi)$.

Thus, the standard translation preserves truth.
Part III

Arthur Prior’s philosophical problem
Prior was a strong adherent of the A-series view

*I believe that what we see as a progress of events is a progress of events, a coming to pass of one thing after another, and not just a timeless tapestry with everything stuck there for good and all...*
Prior’s criticisms of first-order earlier-later logic (B-logic)

Cannot express properly that something presently ("now") is the case.

Cannot capture tenses properly.

Undesired ontological import: The objective existence of instants. (Since first-order variables explicitly refer to instants.)
Prior considered instants to be “artificial” entities which due to their abstractness should not be taken as primitive concepts.

... my desire to sweep ‘instants’ under the metaphysical table is not prompted by any worries about their punctual or dimensionless character but purely by their abstractness.

... ‘instants’ as literal objects, ... go along with a picture of ‘time’ as a literal object, a sort of snake which either eats its tail or doesn’t, either has ends or doesn’t, either is made of separate segments or isn’t; and this picture I think we must drop.
Prior had similar objections to possible worlds

Possible worlds should not be taken as primitive concepts.

...possible worlds, in the sense of possible states of affairs, are not really individuals...

To be the case in a possible world is to be possibly the case; to be the case in an imagined world is to be imagined to be the case; to be the case in a former world is to have been the case...

I take instants, in short, with the same grain of salt as I take possible worlds...
Prior’s preferences

Given the undesired ontological import of first-order earlier-later logic, Prior preferred tense logic.

Some of us at least would prefer to see ‘instants’, and the ‘time-series’ which they are supposed to constitute, as mere logical constructions out of tensed facts.

And similarly, he preferred modal logic.

We understand ’truth in states of affairs’ because we understand ’necessarily’; not vice versa.
So Prior wanted A-logic to encompass B-logic

Prior’s wanted to show that tense logic can be considered as encompassing first-order earlier-later logic.

**Prior’s problem: Ordinary tense logic has much weaker expressive power than first-order earlier-later logic.**

Therefore Prior’s goal was to extend tense logic such that it gets full first-order expressive power.

In technical terms, the goal was to extend tense logic such that first-order earlier-later logic could be translated into it.
What is the deficiency of ordinary tense logic?

Cannot formalize statements involving reference to particular times. An example is

\[
\text{it is five o’clock 10 May 2007}
\]

which is true at some particular time, but false at all other times.
What is the deficiency of ordinary tense logic?

Cannot formalize statements involving reference to particular times. An example is

\[ \text{it is five o'clock 10 May 2007} \]

which is true at some particular time, but false at all other times.

Another example is

\[ \text{at five o'clock 10 May 2007, it is raining} \]

which is about what happened at a particular time.
Part IV

Prior’s solution: Hybrid tense logic
Extend tense logic with hybrid-logical machinery

Enables us to formalize the example statements referring to particular times.

Solves Prior’s philosophical problem (at least from a technical point of view).

In fact, it remedies a number of general deficiencies of ordinary tense and modal logic!
Extension number one:

Add a second sort of propositional symbols

\(a, b, c, \ldots\)

called *nominals*.

Each nominal is true at exactly one moment, thus, a nominal refers to a moment.
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Each nominal is true at exactly one moment, thus, a nominal refers to a moment.

A nominal can be used to formalize the example statement

> *it is five o’clock 10 May 2007.*
Extension number two:

Add operators

@_a, @_b, @_c, \ldots

called *satisfaction operators*.

A formula @_aφ_ is true iff \( \phi \) is true at the moment _a_ refers to.

Remark: Prior used \( T(a, \phi) \) instead of \( @_a\phi \).
Extension number two:

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A formula @_a_\phi is true iff \phi is true at the moment \(a\) refers to.

Remark: Prior used \(T(a,\phi)\) instead of @_a_\phi.

The formula @_a_\phi can be used to formalize the example statement

\[\text{at five o’clock 10 May 2007, it is raining.}\]
Add the *binder* $\forall$.

The formula $\forall a \phi$ is true iff $\phi$ is true whatever moment $a$ refers to.
Formal semantics of hybrid tense logic

An assignment is a function from nominals to moments. We let $g' \sim a g$ mean that $g'$ agrees with $g$ on all nominals save possibly $a$.

The definition of the relation $M, t \models \phi$ is extended with an assignment $g$ and the following clauses are added.

\[
M, g, t \models a \iff t = g(a)
\]
\[
M, g, t \models \@a \phi \iff M, g, g(a) \models \phi
\]
\[
M, g, t \models \forall a \phi \iff \text{for all } g' \text{ such that } g' \sim a g,
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\[\text{it is the case that } M, g', t \models \phi\]
Formal semantics of hybrid tense logic

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\begin{align*}
\mathcal{M}, g, t \models a & \iff t = g(a) \\
\mathcal{M}, g, t \models \Diamond_a \phi & \iff \mathcal{M}, g, g(a) \models \phi \\
\mathcal{M}, g, t \models \forall a \phi & \iff \text{for all } g' \text{ such that } g' \sim^a g, \\
& \quad \text{it is the case that } \mathcal{M}, g', t \models \phi
\end{align*}
\]

Note that the local character of the semantics is not disturbed by nominals and satisfaction operators: Involves only reference to particular moments.

Things are more complicated with the $\forall$ binder.
The extended standard translation to first-order logic

The standard translation $ST_a$ is extended with the following clauses.

$$ST_a(c) = a = c$$
$$ST_a(\Diamond_c \phi) = ST_a(\phi)[c/a]$$
$$ST_a(\forall c \phi) = \forall c ST_a(\phi)$$

Again, $ST_b$ is obtained by exchanging $a$ and $b$. 

Note that a hybrid-logical assignment corresponds to a first-order assignment when nominals are identified with first-order variables. 

Theorem

For any hybrid-logical formula $\phi$ in which the nominals $a$ and $b$ do not occur, $M, g, g(a) \models \phi$ if and only if $M, g \models ST_a(\phi)$. 

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First-order logic can be translated back into hybrid logic

The translation $HT$ is defined by induction.

$$
HT(p^*(a)) = \Diamond_a p \\
HT(a < c) = \Diamond_a Fc \\
HT(a = c) = \Diamond_a c \\
HT(\phi \land \psi) = HT(\phi) \land HT(\psi) \\
HT(\neg \phi) = \neg HT(\psi) \\
HT(\forall a \phi) = \forall a HT(\phi)
$$

Theorem
For any first-order formula $\phi$, it is the case that $M, g \models \phi$ if and only if $M, g, t \models HT(\phi)$ (note that the time $t$ is arbitrary).

Thus, hybrid tense logic (including $\forall$) has the same expressive power as first-order earlier-later logic.

So Prior clearly reached his technical goal.
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So Prior clearly reached his technical goal.
Did Prior reach his philosophical goal?
Prior’s philosophical goal

Prior found that he reached his philosophical goal, namely that of avoiding an ontology including instants.

*The ‘entities’ which we ‘countenance’ in our ‘ontology’ ... depend on what variables we take seriously as individual variables in a first-order theory, i.e. as subjects of predicates ...*
Prior’s philosophical goal, continued

If we prefer to handle instant-variables, . . . as subjects of predicates, then we may be taken to believe in the existence of instants, . . .

If, on the other hand, we prefer to treat either of these as propositional variables, . . . then we may be taken as not believing in the existence of instants, etc. (they don’t exist; rather, they are or are not the case).

However, it has been debated whether or not Prior managed to avoid an instant ontology.
First criticism of Prior’s solution

Binding with $\forall$ is analogous to first-order quantification.

Hence, the ontological import is not reduced by going from first-order earlier-later logic to hybrid tense logic.

Note: This criticism presupposes that nominals are considered as symbols that refer to something, analogous to first-order variables.
Prior’s two views on nominals

We might . . . equate the instant a with a conjunction of all those propositions which would ordinarily be said to be true at that instant, or we might equate it with some proposition which would ordinarily be said to be true at that instant only, and so could serve as an index of it.

The second half of the sentence is in line with considering a nominal as a symbol that refers to an instant.

In the first half of the sentence, a nominal is viewed as a description of the content of an instant.

Remark: Similar to a maximal consistent set.
A criticism from the perspective of contemporary model-theory has been raised by Patrick Blackburn.

*If the fundamental unit of logical modeling is a formal language together with a set-theoretical interpretation, then it makes little sense to claim, for example, that first-order logic automatically brings greater ontological commitment than (say) propositional modal logic.*

*Under the model-theoretic conception, both make use of the same set-theoretic structures, so their ontological commitments are at least prima facie identical.*
Second criticism of Prior’s solution, continued

Perhaps arguments could be mounted (based, perhaps, on the fact that modal logic is decidable and has the finite model property) that modal logic commits us to less. But such arguments would have to be carefully constructed.

In the light of modern correspondence theory, simple knockdown arguments based on the presence or absence of explicit quantifiers in the object language are unconvincing.
Another criticism raised by Blackburn concerns *egocentric logic* which is obtained by

- replacing instants by persons; and
- replacing the earlier-later relation by the shorter-taller relation.

Egocentric logic is just an instance of a general relationship: Any first-order logic has a modal-logical counterpart.

Thus, there is nothing special about tense logic.
But Prior considered tense logic to have a privileged status that distinguishes it from other logics, in particular egocentric logic.

*Tense logic is . . . metaphysically fundamental, and not just an artificially torn-off fragment of the first-order theory of the earlier-later relation.*

*Egocentric logic is a different matter; I find it hard to believe that individuals really are just propositions of a certain sort, . . .*
Third criticism of Prior’s solution, continued

Thus, Prior considered tense logic to have a special status, but there is nothing special about tense logic. Prior concluded:

So far as I can see, there is nothing philosophically disreputable in saying that

(i) persons just are genuine individuals, so that their figuring as individual variables in a first-order theory needs no explaining . . .

(ii) instants are not genuine individuals, so that their figuring as values of individual variables does need explaining, and it is the related ‘modal’ logic (tense logic) which gives the first-order theory what sense it has.

However, Blackburn criticizes Prior’s conclusion for being unsatisfactorily justified.
Part VI

The development of hybrid logic since Prior
Publication of the paper "An Approach to Tense Logic"

Hybrid tense logic extended with a third "course of history" sort (branches in trees)

Bull’s idea can be generalized: Restrict the interpretation of propositional symbols corresponding to different sorts of information.

Another example: Sort for intervals (convex sets)

The general idea of sorting is almost unexplored!
The Sofia School (1980s and early 1990s)

Solomon Passy and Tinko Tinchev reinvented Prior’s hybrid-logical machinery (in the context of Propositional Dynamic Logic).

Valentin Goranko invented what is denoted the ↓ binder.
Contemporary hybrid logic: Proof-theory

Axiomatics:

Completeness results (Patrick Blackburn et al.)
Notable property: Completeness of axiom systems is preserved when extended with pure formulas
(formulas where all propositional symbols are nominals).
Contemporary hybrid logic: Proof-theory

Axiomatics:

Completeness results (Patrick Blackburn et al.)
Notable property: Completeness of axiom systems is preserved when extended with *pure* formulas
(formulas where all propositional symbols are nominals).

Analytic proof-theory:

Tableau systems (Blackburn, Bolander, Braüner, Marx, Hansen)
Also extendable with pure formulas

Natural deduction and Gentzen systems (Braüner, Seligman)
Extendable with rules for *geometric theories* 
($\forall \bar{a}(\phi \rightarrow \psi)$ where $\phi$ and $\psi$ are built using $\bot$, $\land$, $\lor$, and $\exists$)

This uniformity is NOT shared by standard modal-logical proof-theory!
...a deductive treatment congenial to modal logic is yet to be found, for Hilbert systems are not suited for the purpose of actual deduction, ... 

...only exceptional systems ...seem to be characterizable in terms of reasonably simple rules.
Proof-theoretical conclusion

The deficiency of standard analytic proof-theory for modal logic is remedied by hybridization!
Contemporary hybrid logic: Many other important issues

- Model theory
- Decidability and complexity
- Many-valued and intuitionistic hybrid logic
- Epistemic hybrid logic
- Relations with other fields, in particular description logics
- Theorem provers and other computational tools
More information

Areces and ten Cate’s chapter in the *Handbook of Modal Logic*, Elsevier, 2007

Braüner’s chapter in the *Handbook of Philosophical Logic*, volume 16, Springer, to appear